Optimal-area Visibility Representations of Outer-1-plane Graphs

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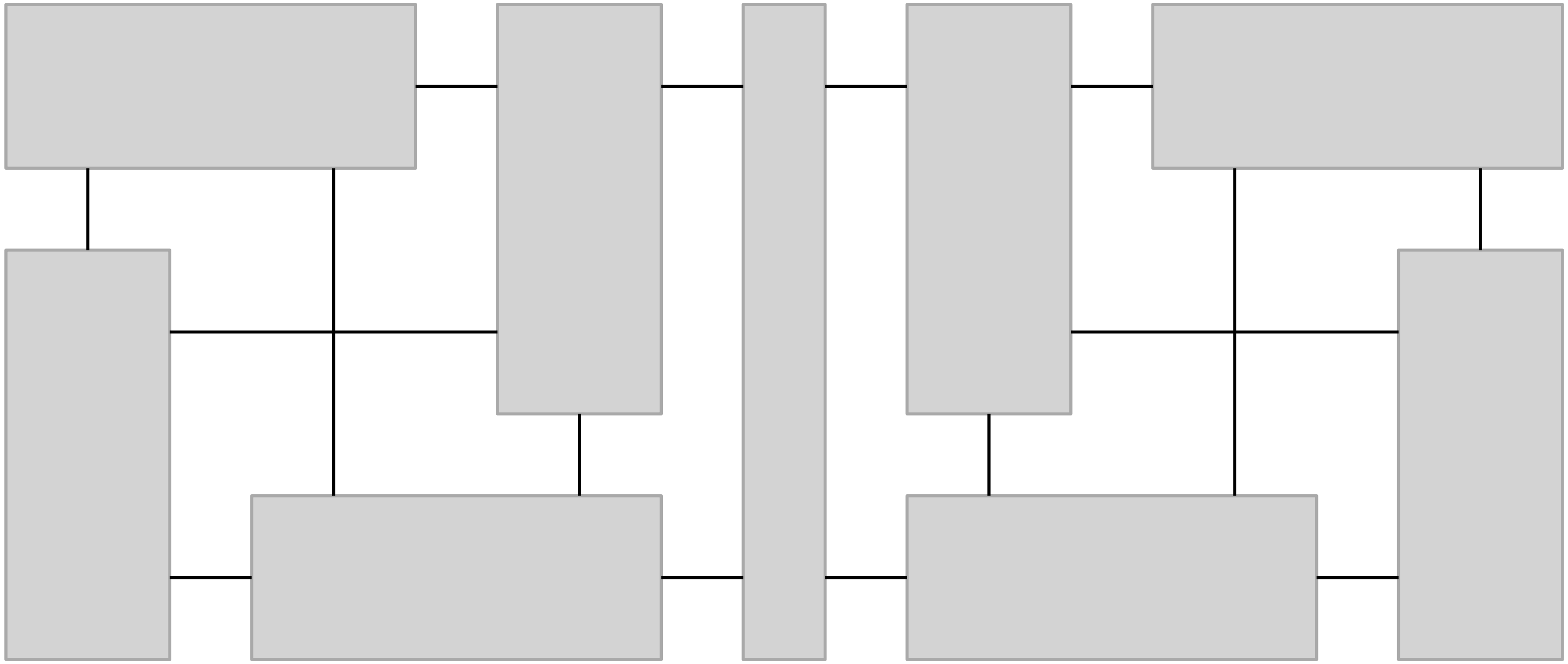
Visibility Representations



Vertices = axis-aligned rectangles
Edges = axis-aligned segments, called visibilities

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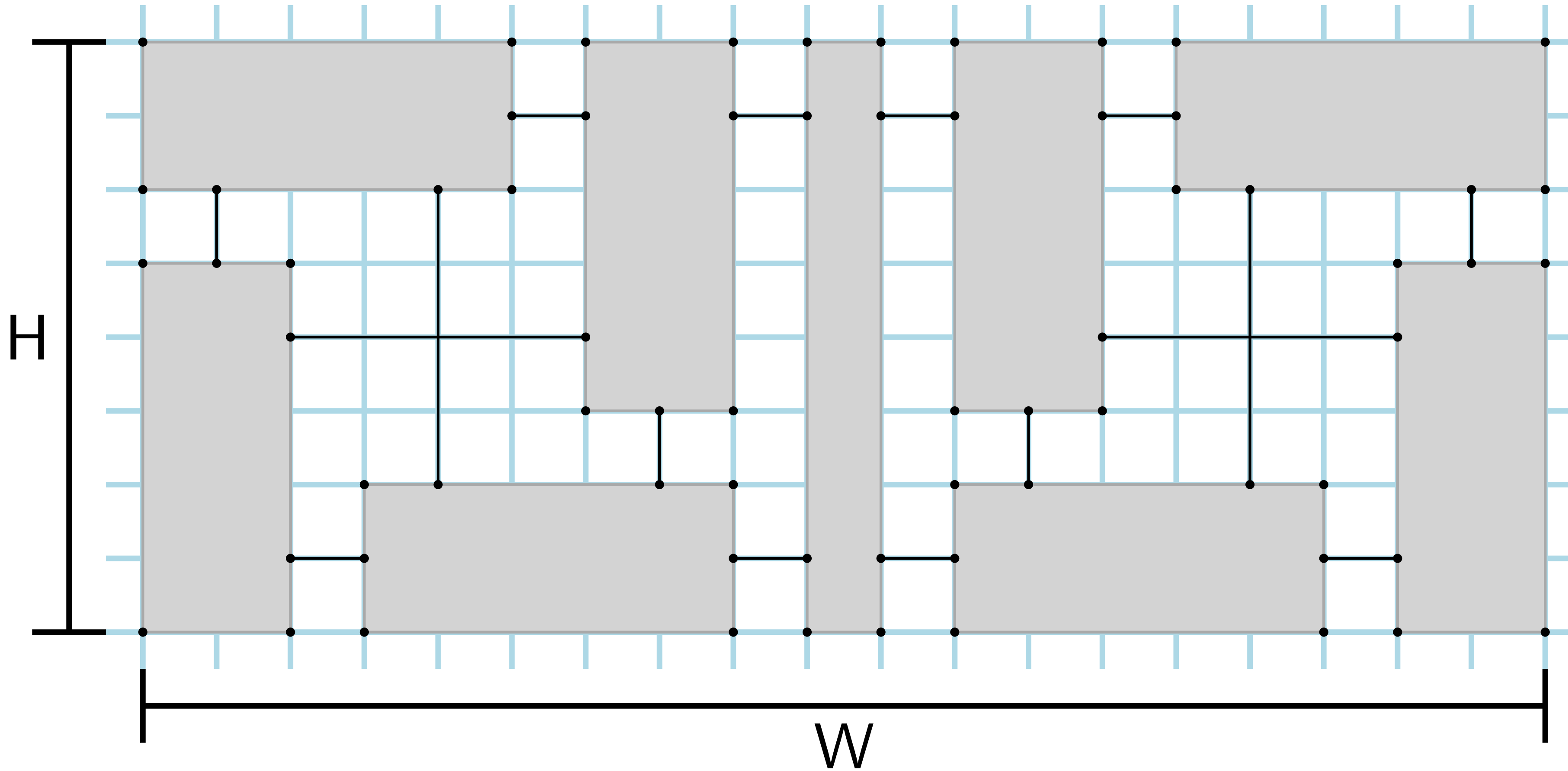

Vertices = axis-aligned rectangles
Edges = axis-aligned segments, called visibilities

Integer Grid = vertex corners + vertex-edge attachment points have integer coordinates

Area = Width × Height

Existence & Area Bounds

Not all graphs admit a VR

- Recognition is NP-hard [Shermer, 1996]
- All planar graphs admit a VR [Otten & van Wijk.,1978]

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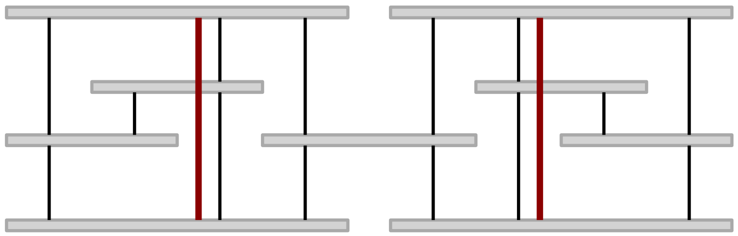
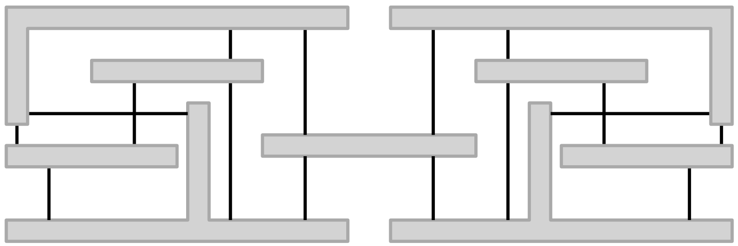
If a VR exists, how small can the grid be?

- $O((n+m) \times (n+m)) = O(n \times n)$ area is always sufficient
- Planar graphs may require $\Omega(n^2)$ area [Fößmeier et al., 1997]
- Series-parallel graphs have VRs in $O(n^{1.5})$ area [Biedl, 2013]
- Outerplanar graphs have VRs in $O(n \cdot \log n)$ area [Biedl, 2011]

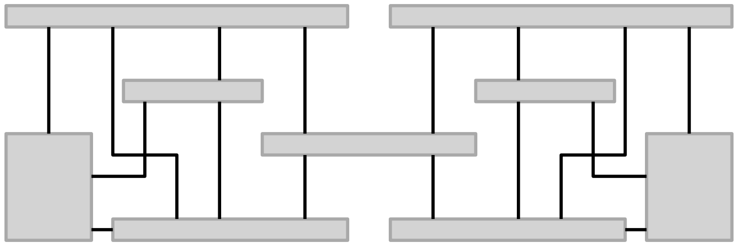
Variations



bar-(j,k) VR



ortho-polygon (OP) VR

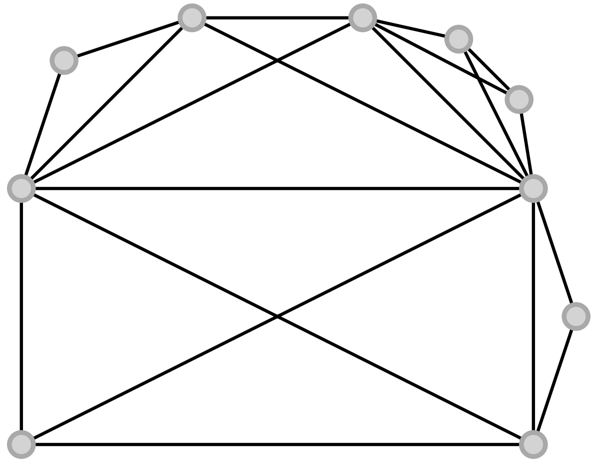


orthogonal box-drawings

bar-(j,k) VR = visibilities can see through vertices
OP VR = vertices are general orthogonal polygons
Orthogonal box-drawings = edges are general orthogonal poly-lines

Optimal-area Visibility Representations of Outer-1-plane Graphs

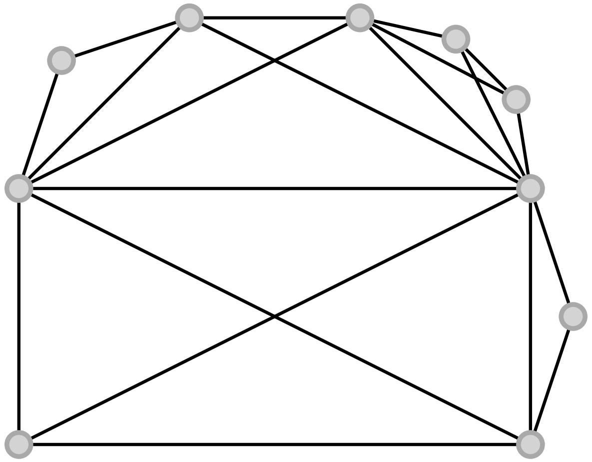
Outer 1-planar Graphs



Outer 1-planar graphs = can be drawn s.t. all vertices are on the boundary of the outer face and each edge is crossed at most once

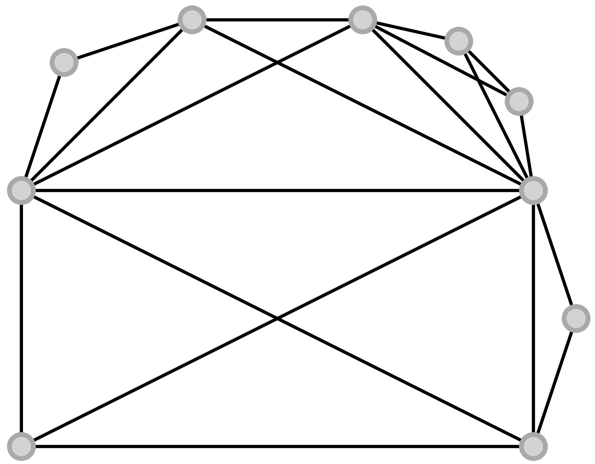
Outer 1-plane graphs = with a fixed outer 1-planar embedding (i.e., fixed rotation scheme and fixed pairs of crossing edges)

Outer 1-planar Graphs



Outer 1-planar graphs:

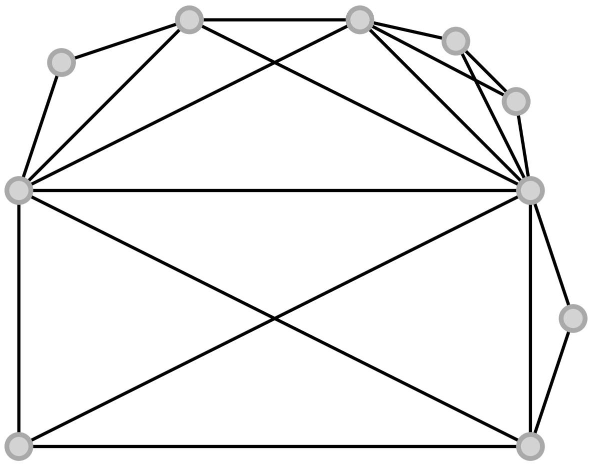
Outer 1-planar Graphs



Outer 1-planar graphs:

Planar and linear time recognition [Auer et al., 2015; Hong et al., 2015]

Outer 1-planar Graphs



Outer 1-planar graphs:

Planar and linear time recognition [Auer et al., 2015; Hong et al., 2015]

May require $\Omega(n^2)$ area in any planar VR [Biedl, 2020]

Admit embedding-preserving orthogonal box-drawings with 2 bends per edge in $O(n \log n)$ area [Biedl, 2020]

Question

Can we always compute a VR of an outer-1-plane graph?

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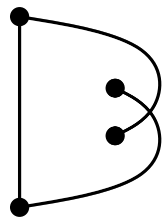
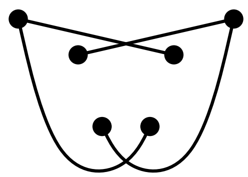
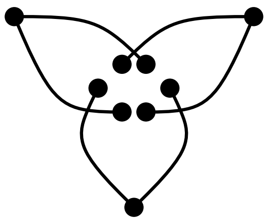
Theorem[Biedl, Liotta, M., 2018].
A 1-plane graph admits a VR if and only if it contains no B-configurations, no W-configurations, and no T-configurations.



B-configuration



W-configuration



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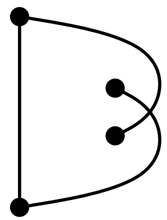

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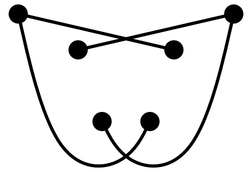

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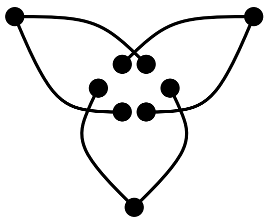

T-configuration

Can we achieve subquadratic area bounds?

Contribution: subquadratic bounds

drawing style	lower bound	upper bound
VR	$\Omega(n^{1.5})$	$O(n^{1.5})$
complexity-1 OP VR	$\Omega(n\,pw(G))$	$O(n^{1.48})$
1-bend orth. box-drawing	$\Omega(n\,pw(G))$	$O(n^{1.48})$

EMBEDDING-PRESERVING

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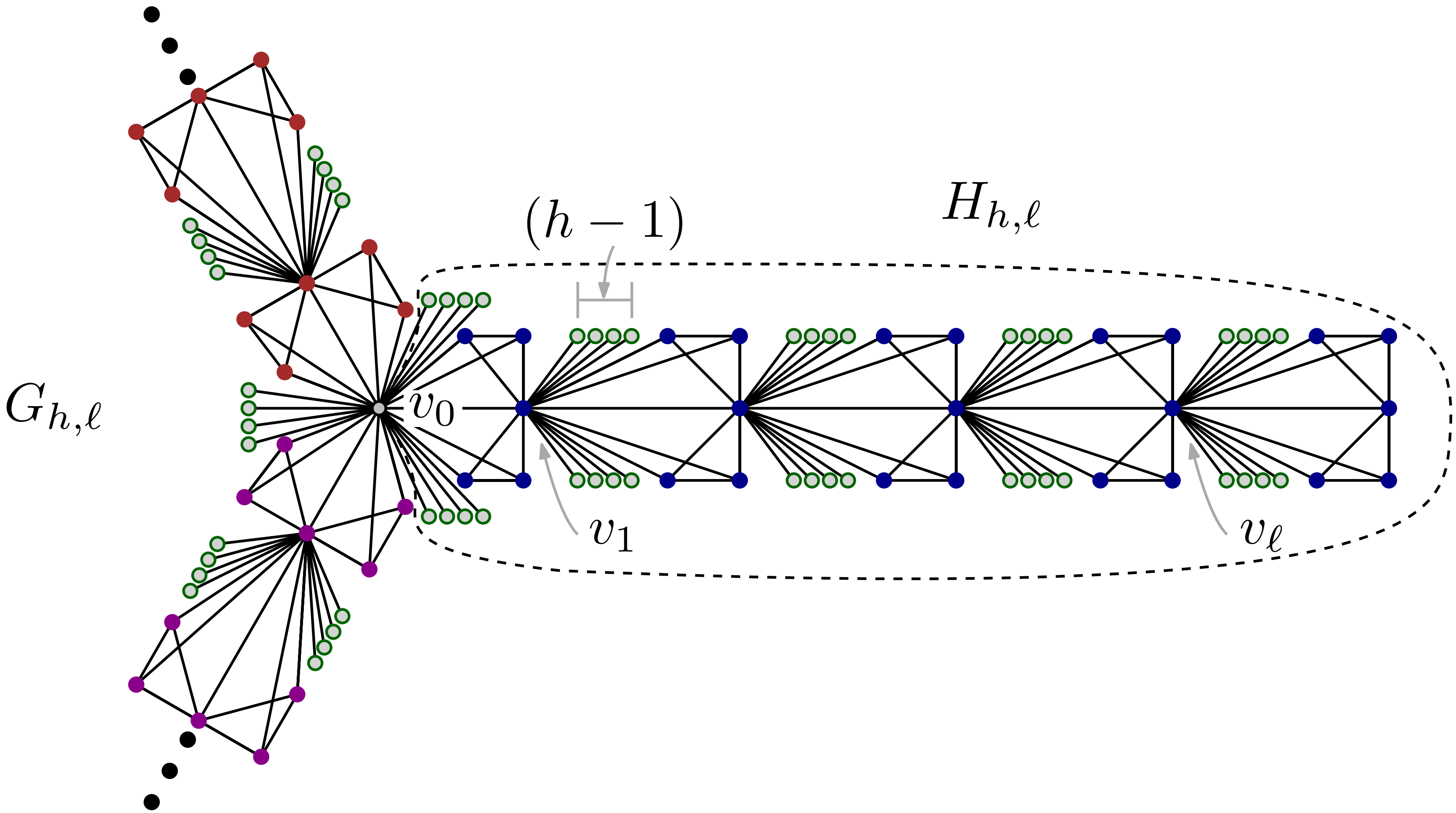
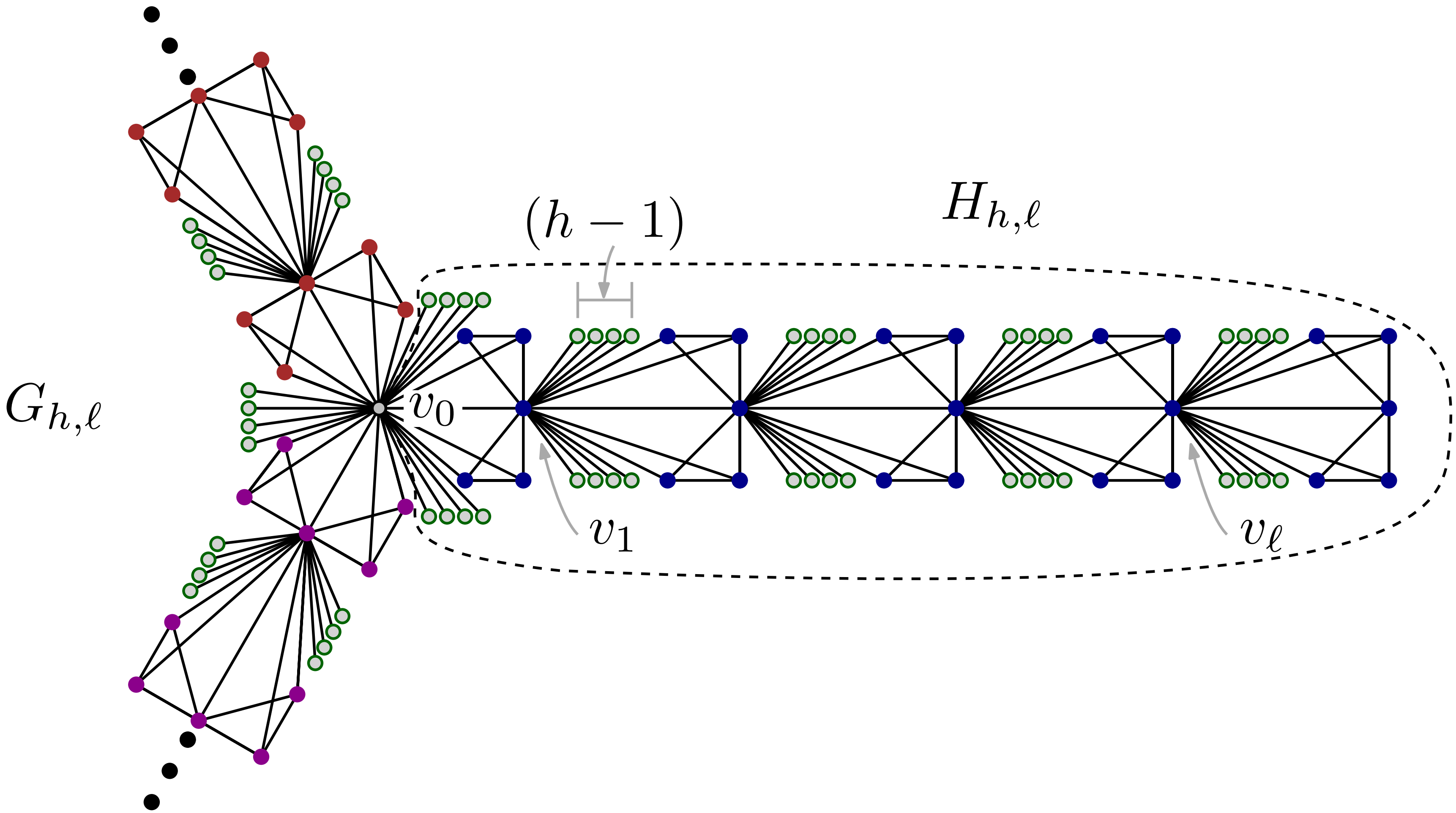


Lower Bound for VRs

Lemma: Any VR Γ of $G_{h,\ell}$ is such that if a rectangle has height at most h, then Γ's width and height are $\Omega(\ell)$



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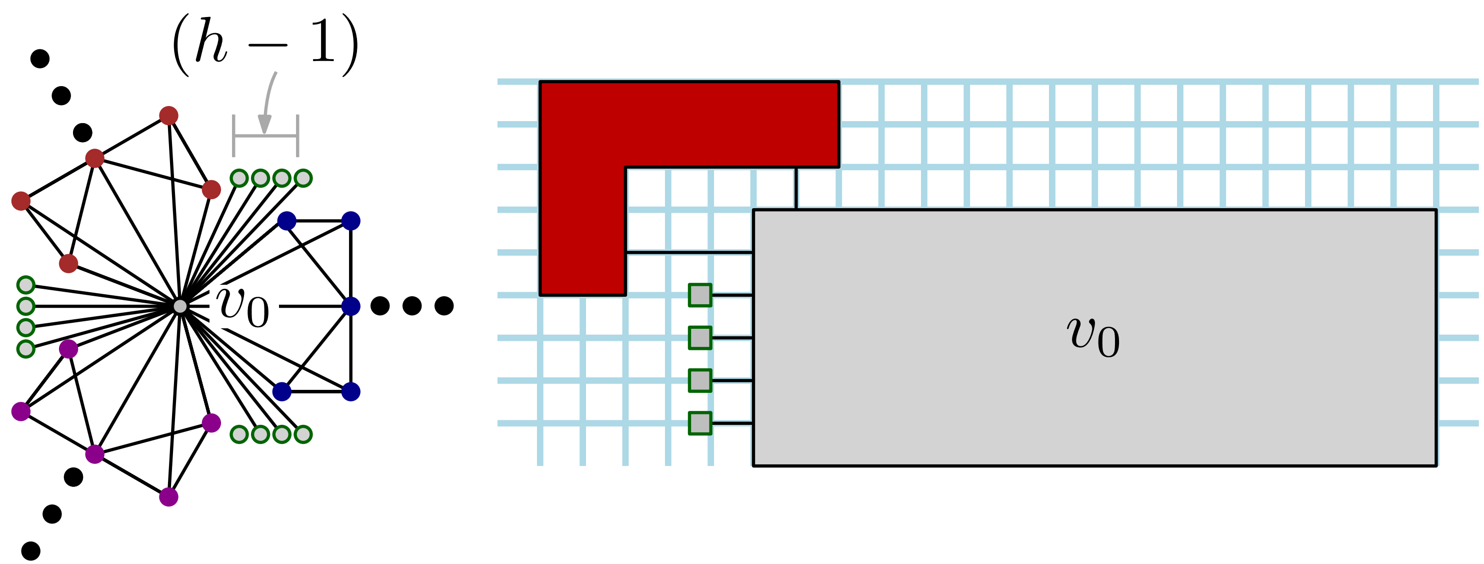

Key observation: In any embedding-preserving VR, there is at most one copy of $H_{h,\ell}$ on the left side and at most one copy on the right side of v_0.
So one copy is such that all edges are vertical and, say, downward.

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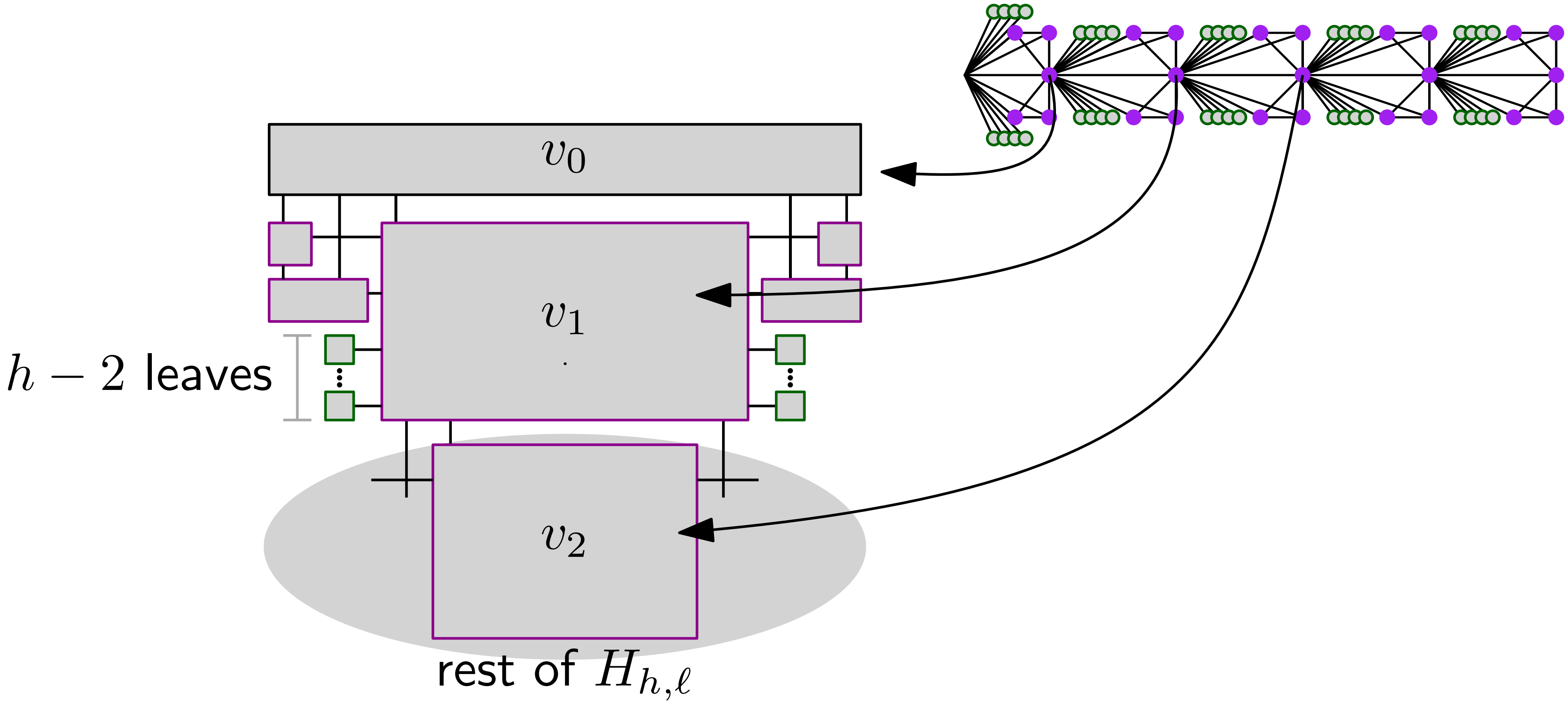

Proof by induction

Lower Bound for VRs

Theorem. For any N there is an n-vertex outer-1-plane graph with $n \geq N$ vertices such that any embedding-preserving VR has area $\Omega(n^{1.5})$

To build G: fix $h = \ell = \lceil \sqrt{N} \rceil$; add N leaves at v_0.

Consider any VR Γ of G.
Since $\deg(v_0) > N$, W (or H) is $\Omega(N)$.

If the height of a rectangle is more than $h = \sqrt{N}$ we are done, else by the previous lemma again the height is $\Omega(\ell) = \Omega(\sqrt{N})$.

Contribution

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EMBEDDING MAY NOT BE PRESERVED

Upper Bound for VRs

Theorem. Every n-vertex outer-1-plane graph has an embedding-preserving VR of area $O(n^{1.5})$.

Upper Bound for VRs

It is enough to show that the height is $O(n^{0.5})$

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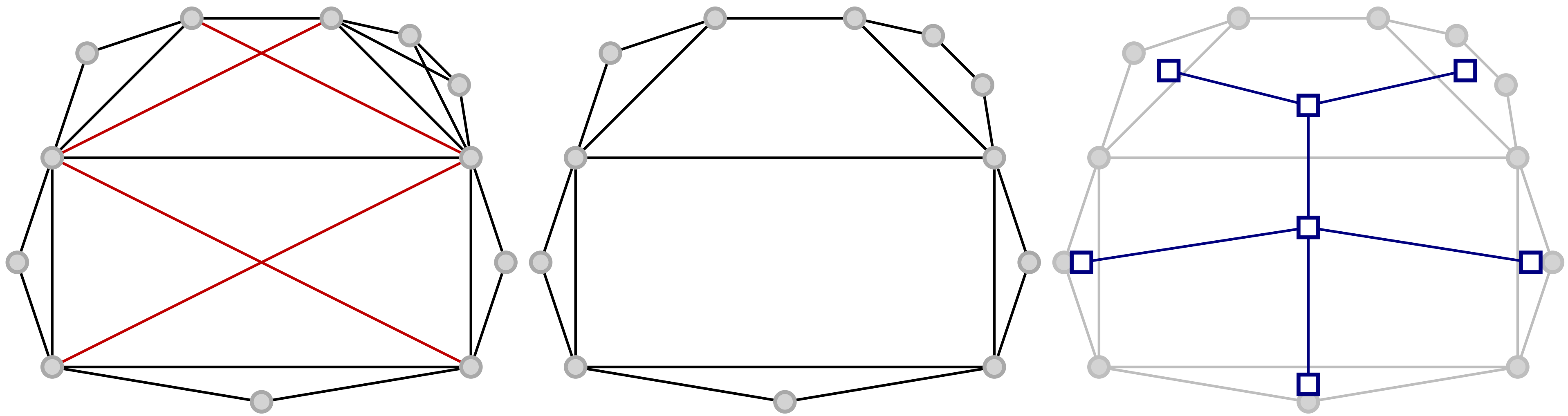
Let G be a maximal-planar outer-1-plane graph
The weak dual $\overline{G^*}$ of the planar skeleton $\overline{G}$ of G is a tree of degree at most four



G $\qquad$ $\overline{G}$ $\qquad$ $\overline{G^*}$

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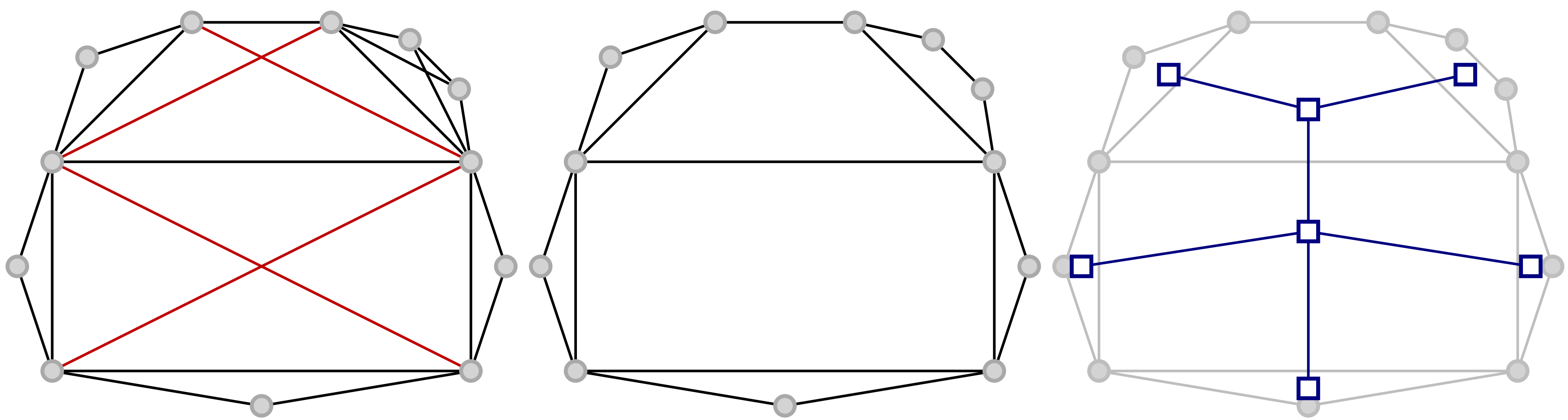
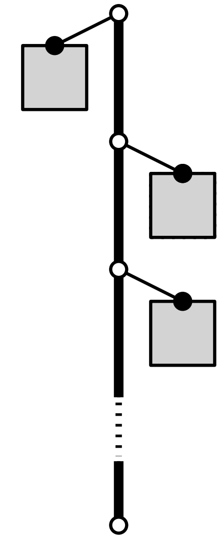


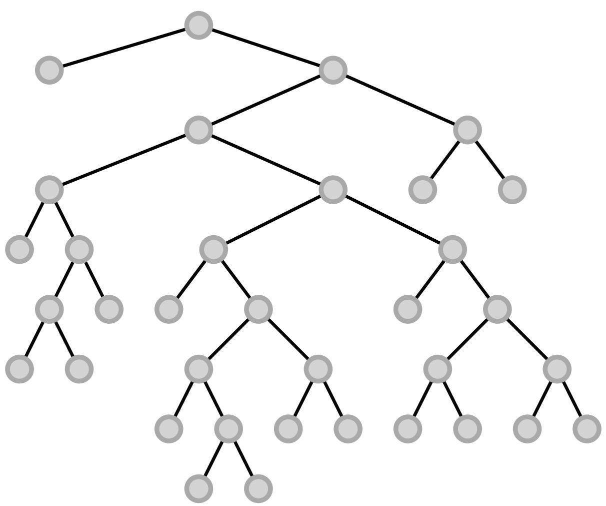
G $\overline{G}$ $\overline{G^*}$

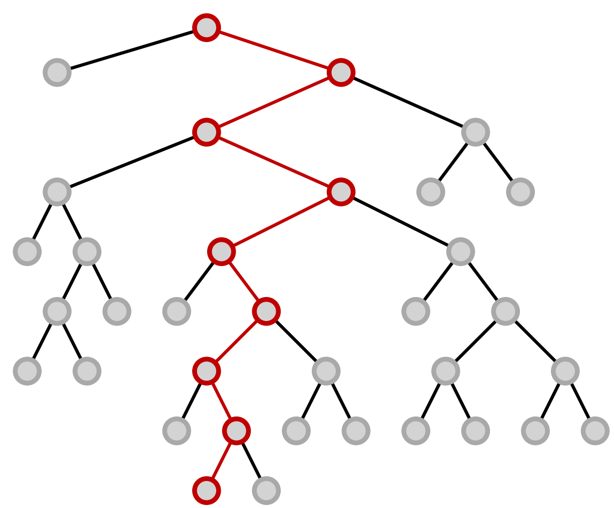
Idea: exploit tools known for so-called LR-drawings of binary trees

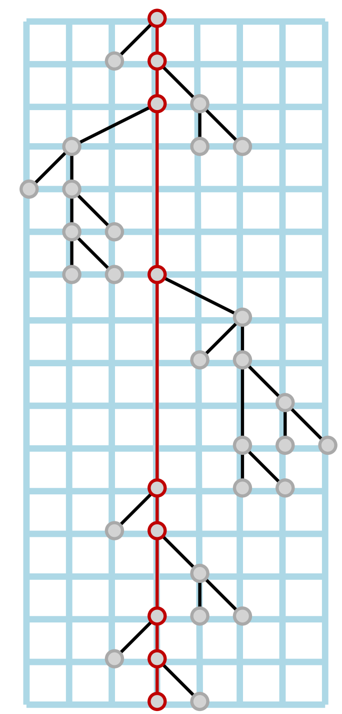
Upper Bound for VRs

Theorem [Chan, 2002]. Let $p = 0.48$. Given any ordered binary rooted tree T of n vertices, there exists a root-to-leaf path π such that for any left subtree α and any right subtree β of π, $|\alpha|^p + |\beta|^p \leq (1 - \delta)n^p$, for some constant $\delta > 0$.









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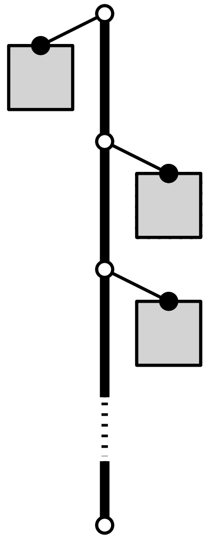
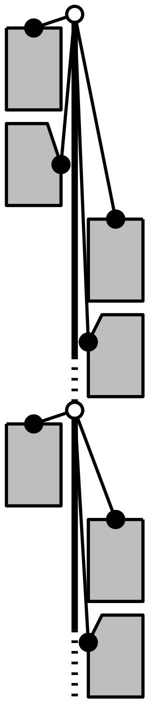




Theorem [Biedl et al., 2021]. Let $p = 0.48$. Given any ordered ~~binary~~ rooted tree T of n vertices, there exists a root-to-leaf path π such that for any left subtree α and any right subtree β of π, $|\alpha|^p + |\beta|^p \leq (1-\delta)n^p$, for some constant $\delta > 0$.

Upper Bound for VRs

High-level plan;

- Pick a path π in $\overline{G^*}$ that satisfies the theorem
- Construct a drawing of height $h(F)$, where F is the size of $\overline{G^*}$, such that

$$h(F) = \max_{|\alpha|^p + |\beta|^p \leq (1-\delta)n^p} \{h(|\alpha|) + h(|\beta|)\} + O(\sqrt{F})$$

One can verify that $h(F) \in O(\sqrt{F}) \in O(\sqrt{n})$

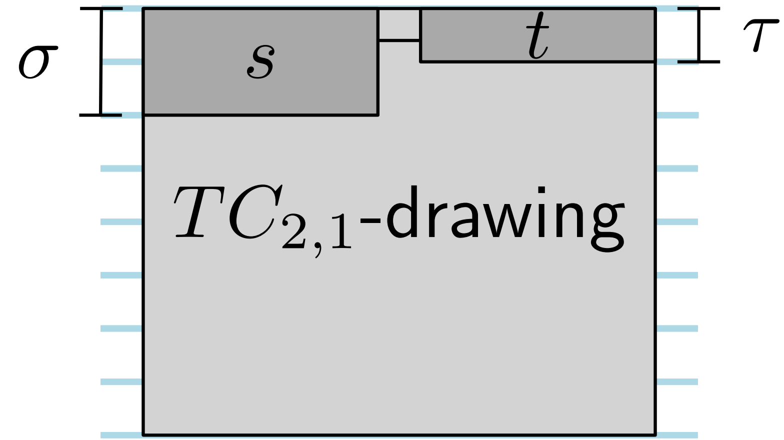
↓

We prove by induction that $h(F) \leq \frac{12}{\delta}\sqrt{F} - 7$

$h(F) = \max_{\alpha,\beta}\{h(|\alpha|) + h(|\beta|)\} + 11\sqrt{F} + 7 \leq$
$\leq \frac{12}{\delta}\sqrt{|\alpha|} + \frac{12}{\delta}\sqrt{|\beta|} + 11\sqrt{F} - 7 \leq$
$\leq \frac{12}{\delta}(1-\delta)^{0.5/p}\sqrt{F} + 11\sqrt{F} - 7 \leq$
$\leq \frac{12}{\delta}(1-\delta)\sqrt{F} + 11\sqrt{F} - 7 =$
$= \frac{12}{\delta}\sqrt{F} - \sqrt{F} - 7 \leq$
$\leq \frac{12}{\delta}\sqrt{F} - 7$

Construction

Lemma. Let G be an outer-1-plane graph. Then it admits an embedding-preserving VR that is a $TC_{\sigma,\tau}$-drawing of height $h(F)$.



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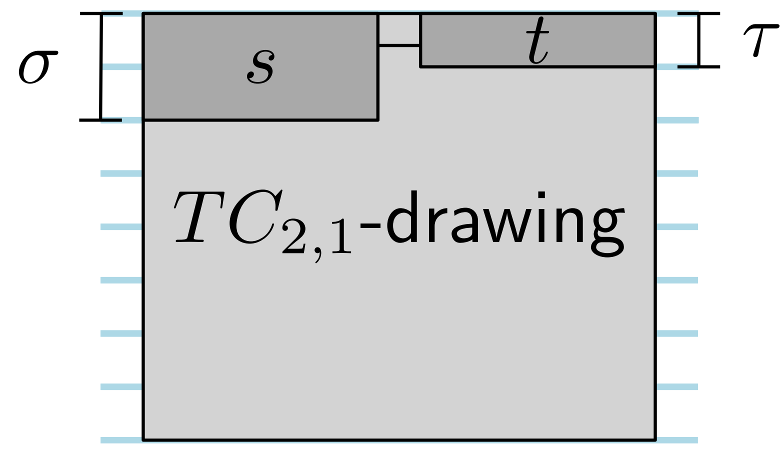
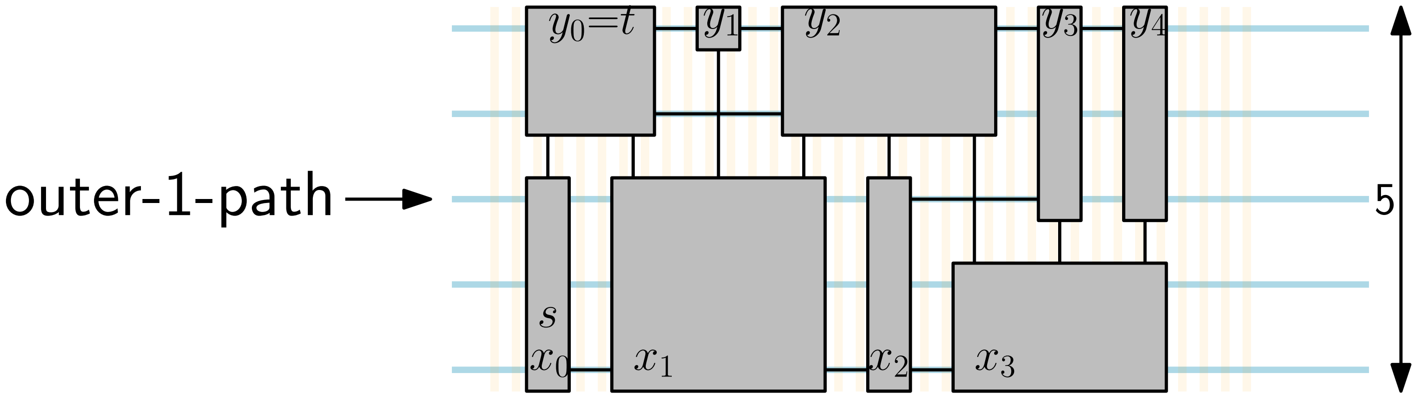


Proof by induction on F
Base case with $F = 1$ and $h(1) = 3$ is trivial

Construction

We first draw straight the primal graph P_π of π



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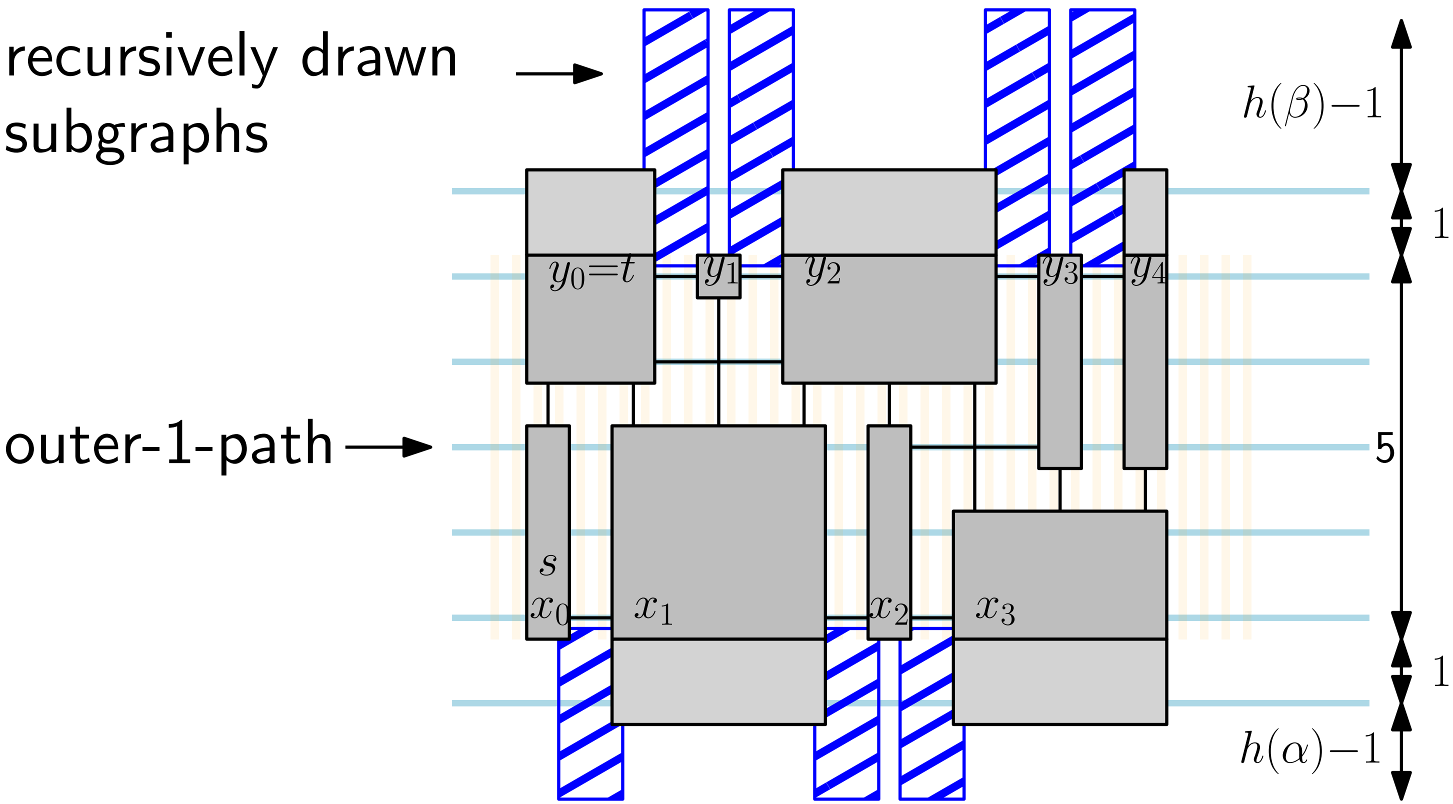
We first draw straight the primal graph P_π of π

We next merge recursively computed drawings of subgraphs hanging at π



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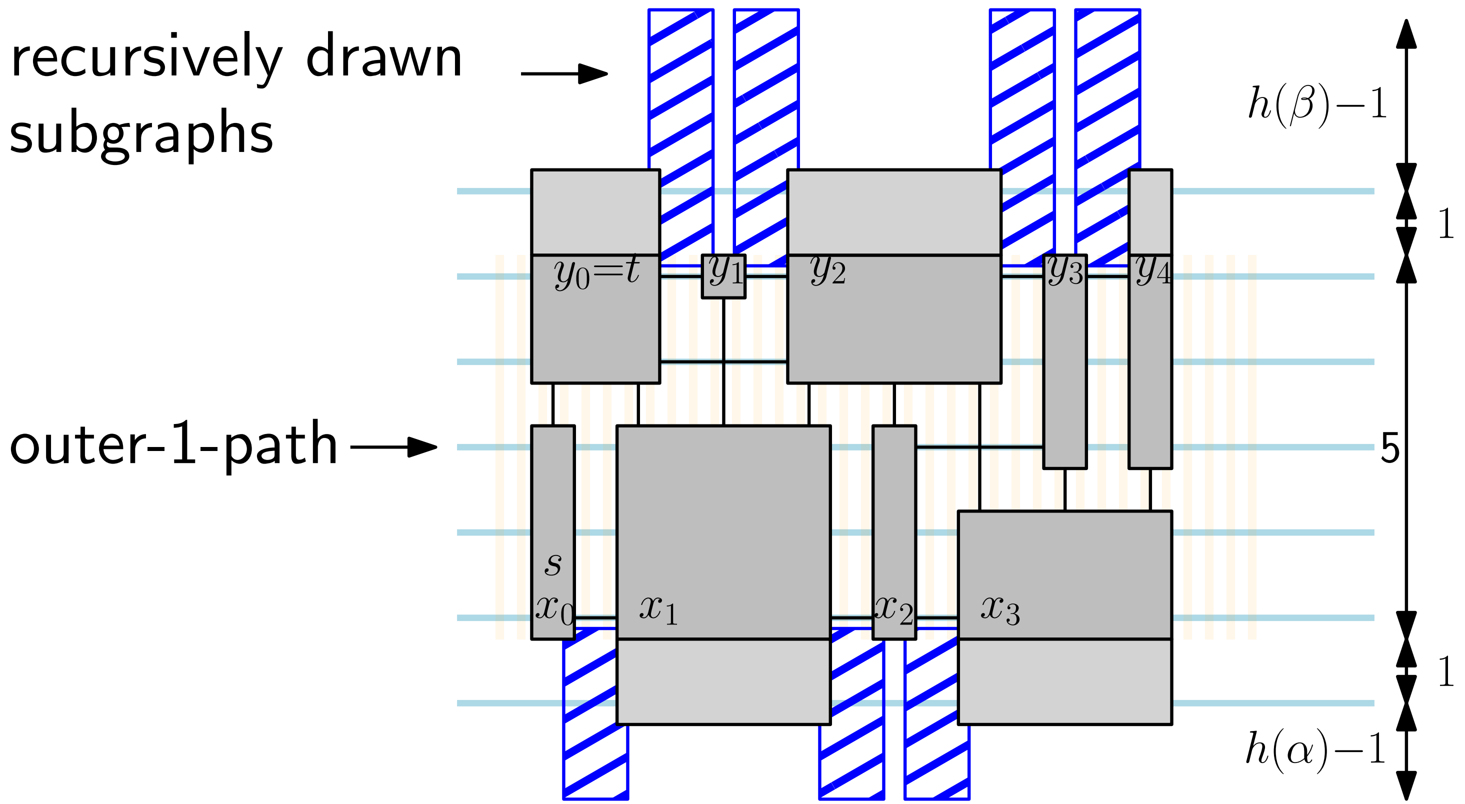
Problem: the drawing is not a $TC_{\sigma,\tau}$-drawing

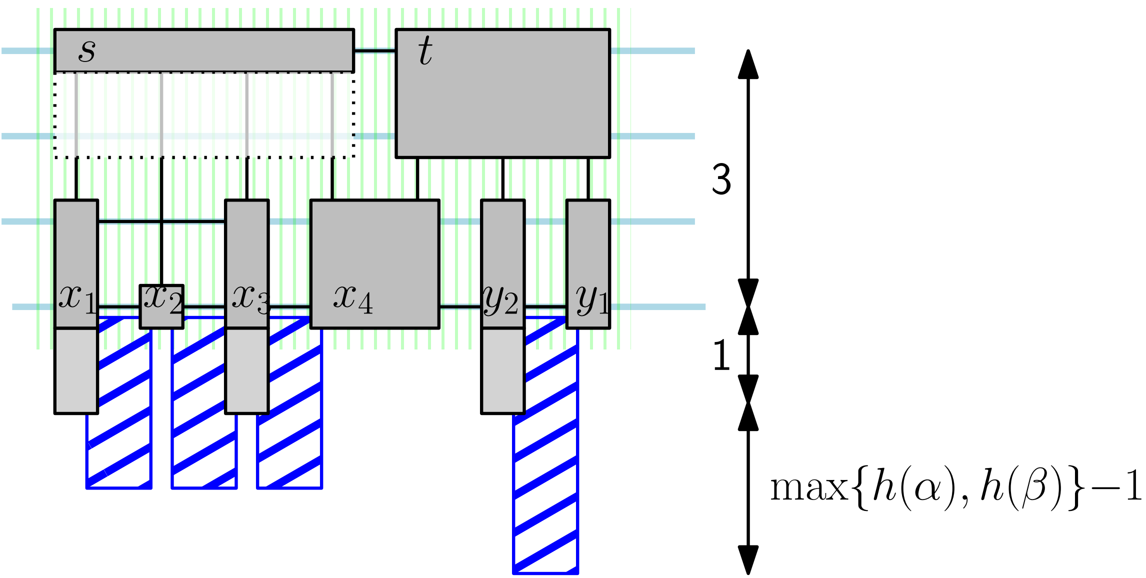


Construction

We further decompose the graph.
The cap is the outer-1-path that contains s, t and all vertices adjacent to s and t.

A $TC_{\sigma,\tau}$-drawing of C and of its hanging subgraphs can easily be computed.

$TC_{1,2}$-drawing of C



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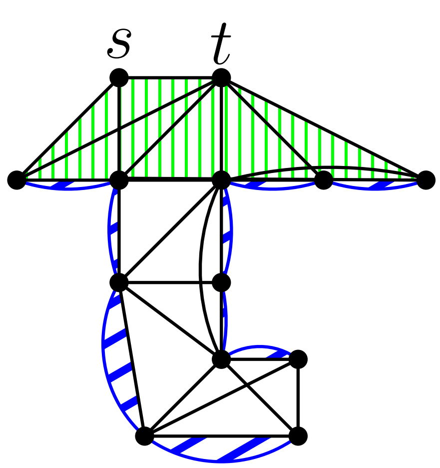
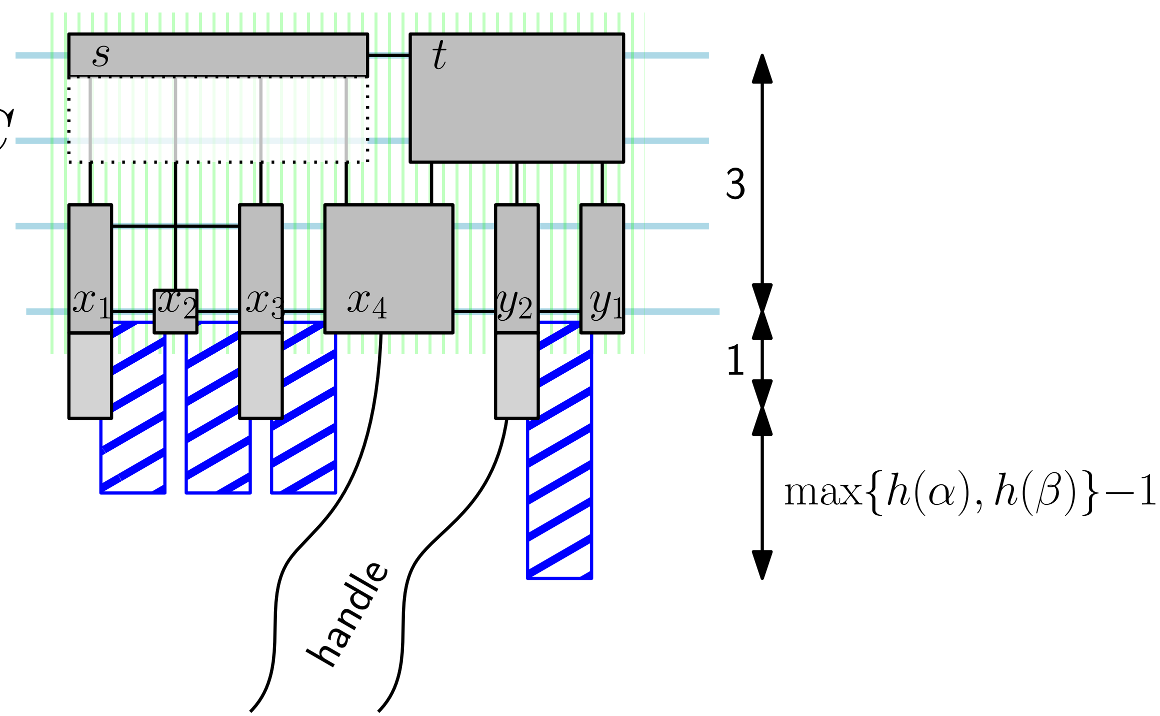
The handle is the part of P_π not in C.

A $TC_{\sigma,\tau}$-drawing of C and of its hanging subgraphs can easily be computed.

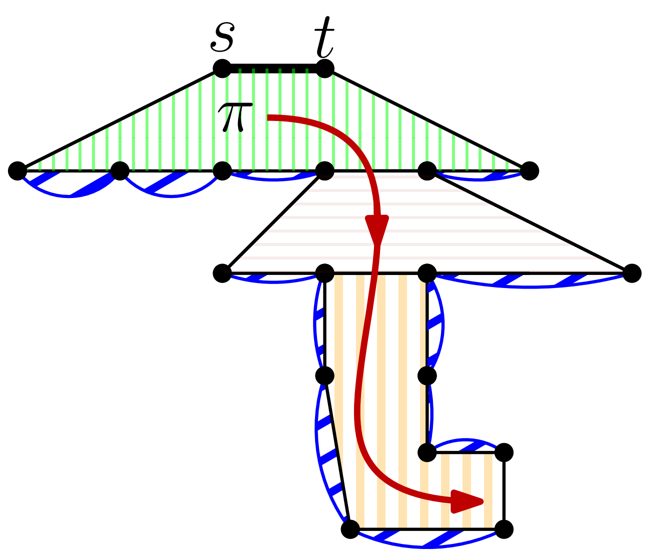
$TC_{1,2}$-drawing of C





Construction

We may need to extract $k > 1$ consecutive caps, for some parameter k.



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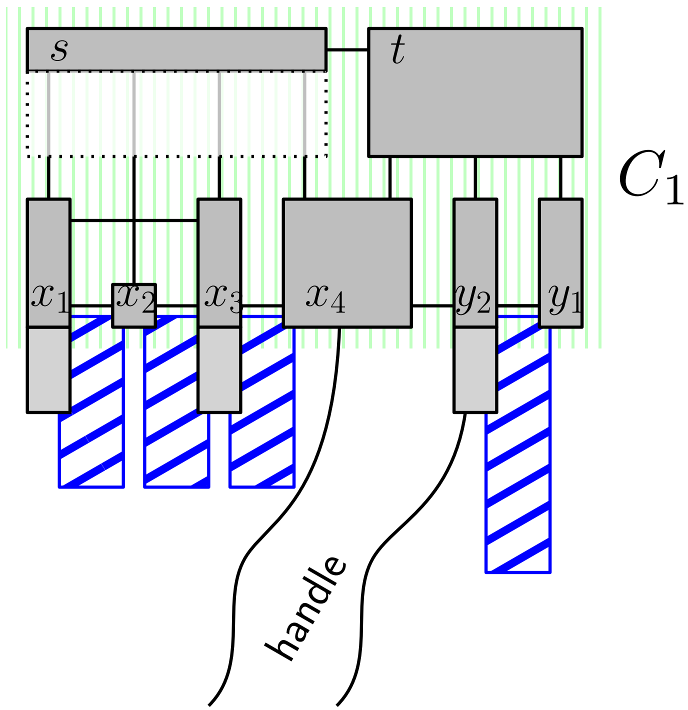

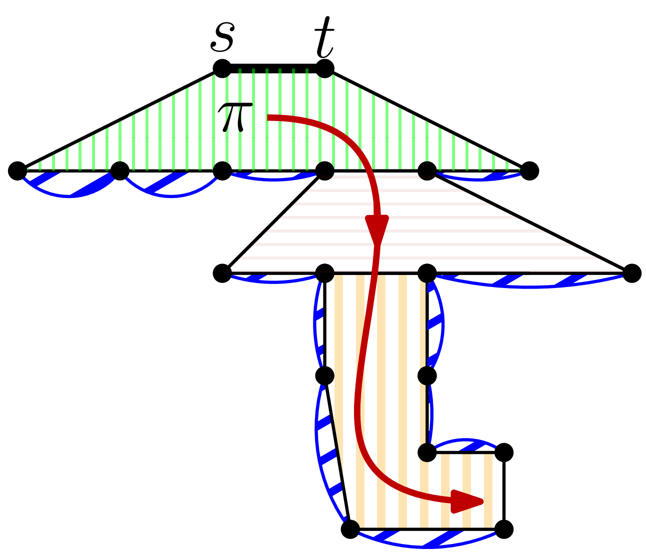

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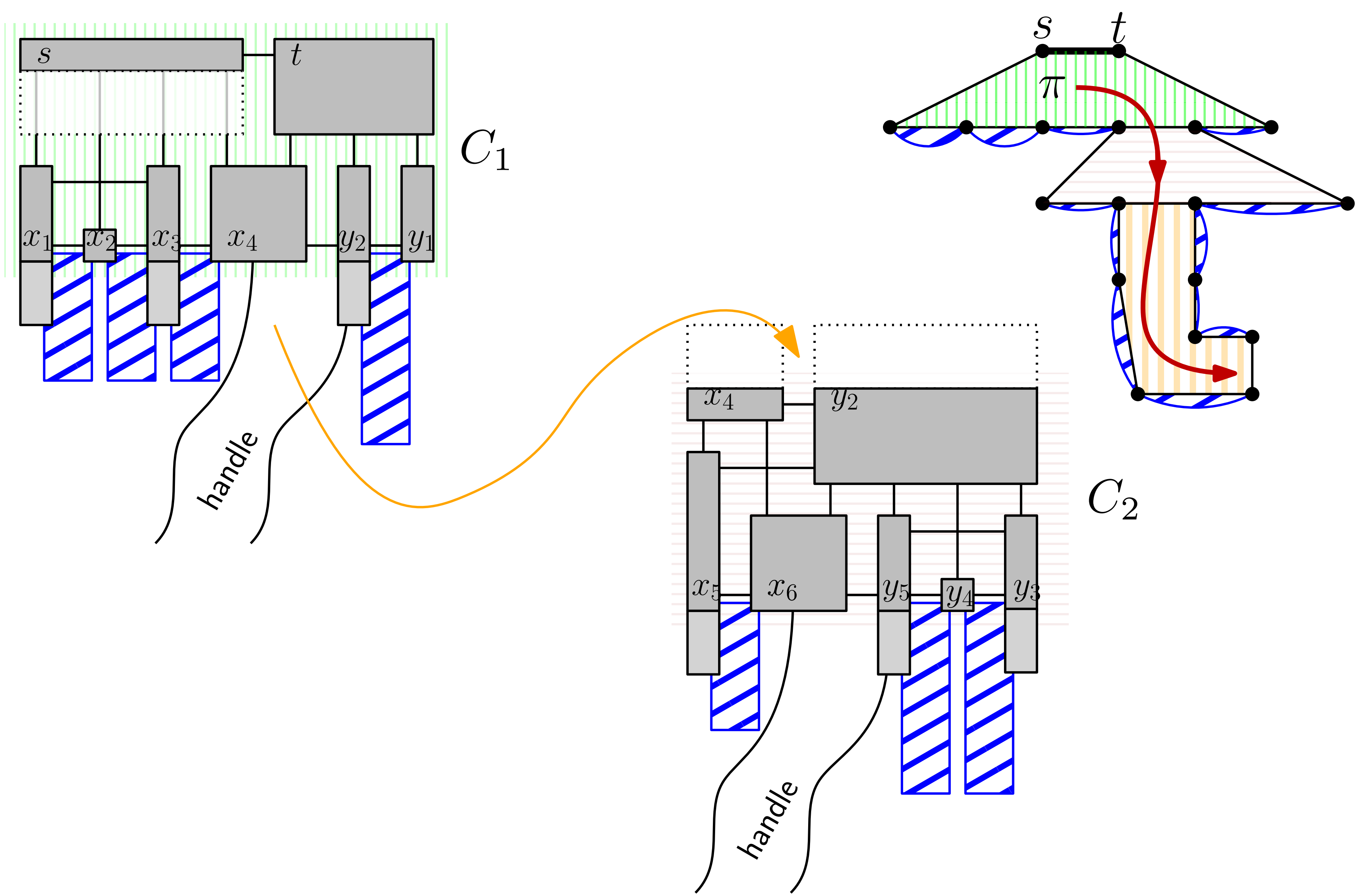
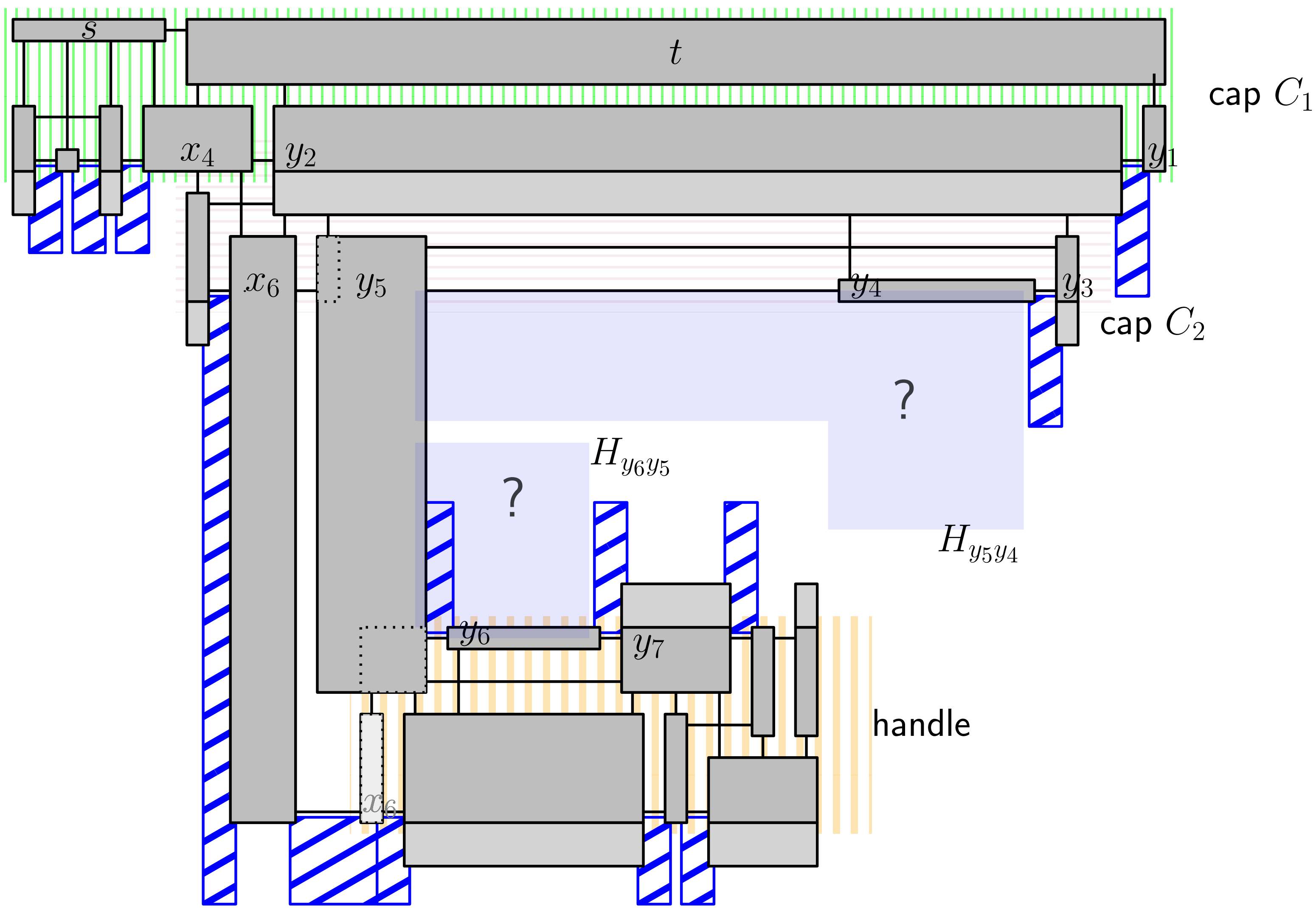


Construction

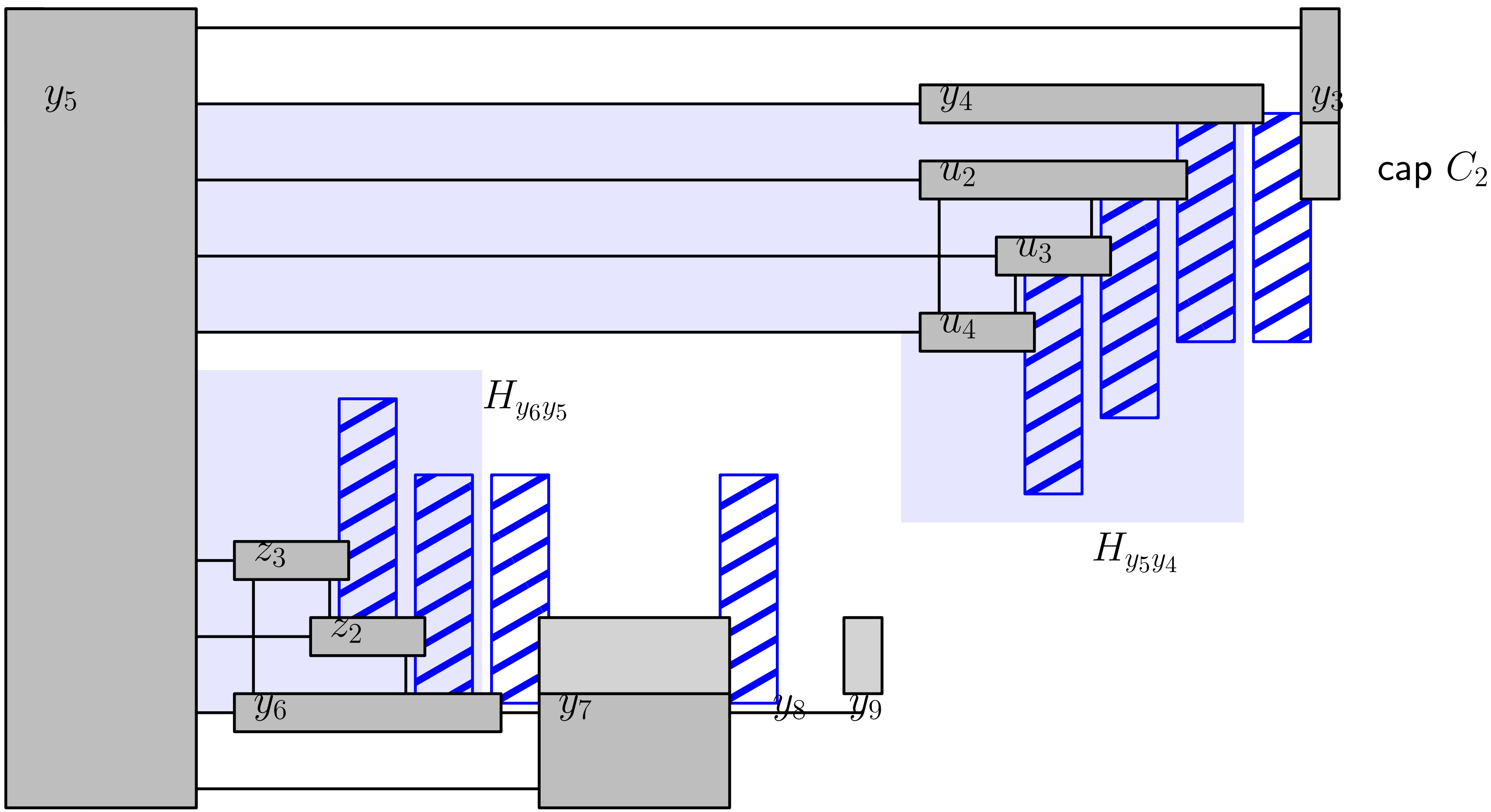
Patching together the drawing of the cap(s) together with a drawing of the handle is the main challenge.



Construction

We need an ad-hoc construction that requires a number of extra rows that depends on the maximum number of edges (D) used to attach these special subgraphs.

One can show that the height is then $h(|\alpha|) + h(|\beta|) + 3k + D + 4 \leq h(F)$ by choosing $k \leq \sqrt{n} + 1$.



Open Problems

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- Can we achieve $O(n^{1+\varepsilon})$ area by using newer results on star-shaped drawings [Frati et al., 2020]?
- Can we achieve $o(n^{1.48})$ area for OP VRs of complexity 2 or 3?
- Can we extend some of our results to other subclasses of 1-planar graphs?

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THANKS FOR YOUR ATTENTION!